ERGODIC DECOMPOSITION OF A TOPOLOGICAL SPACE

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ABSTRACT
Given a positive contraction, P, on $C(X)$ we define the conservative and dissipative parts of P and establish divergence of $\Sigma P^n f(x)$ on the conservative part of X.

Let X be a locally compact normal space which is also σ compact: $X = \bigcup X_n$ where X_n is compact. In order to avoid topological difficulties we shall assume also that every Borel set is a Baire set.

Let P be an operator on the space of real valued, bounded and continuous functions, $C(X)$, such that:

(1) $$P1 \leqq 1 \text{ and if } f \geqq 0 \text{ then } Pf \geqq 0.$$

Every functional on $C(X)$ is given by a regular, bounded, finitely additive measure: $x^*f = \int f d\mu$. Every such measure is σ additive on every compact set. See [1, IV. 6.2 and III. 5.13].

Let us assume, in addition to (1), that

(2) *If* $0 \leqq \mu$ *is a* σ *additive regular measure then so is* $P^*\mu$.

In particular $P^*\delta_x$ is σ additive where δ_x is a unit mass at x. Define

(3) $$P(x, A) = P^*\delta_x(A).$$

then $P(x, \cdot)$ is a σ additive positive and regular measure, on the Borel sets, with $P(x, X) \leqq 1$. Now, if $f \in C(x)$ then $Pf(x) = P^*\delta_x(f) = \int P(x, dy)f(y) \in C(x)$ and is thus measurable. The set of functions f such that $\int P(x, dy)f(y)$ is measurable, is an additive and monotone class and thus includes every bounded Borel function. Thus:

(4) $P(\cdot\, ; A)$ *is measurable for every Borel set* A.

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If $0 \leqq f$ *is measurable the operator* P *extends to* f *by* $Pf(x) = \int P(x, dy) f(y)$ *and* Pf *is again measurable.*

Note that (2) was used for $\mu = \delta_x$ only.

In particular Pf is defined for every non-negative lower semi-continuous function f. By [3, page 176] Pf is again lower semi-continuous. Thus

(5) *Let* $\mathfrak{A} = \{f \mid 0 \leqq f \leqq 1$ *and* f *is lower semi continuous*$\}$ *then* $P\mathfrak{A} \subset \mathfrak{A}$.

In order to define the dissipative part of X we shall use the class $\mathfrak{A}_1$:

$$\mathfrak{A}_1 = \{f \mid f \in \mathfrak{A},\ Pf \leqq f \text{ and } \lim P^n f(x) = 0 \text{ for every } x \in X\}. \tag{6}$$

The class $\mathfrak{A}_1$ is not empty since $0 \in \mathfrak{A}_1$. The dissipative part of X will be denoted by W:

$$W = \{W_f \mid f \in \mathfrak{A}_1\} \text{ where } W_f = \{x \mid f(x) > 0\}. \tag{7}$$

Note that W_f is an open set and so is W.

Lemma 1. *Let* K *be a compact set in* W. *There exists a function* $f \in \mathfrak{A}_1$ *such that* $f(x) \geqq \delta > 0$ $x \in K$.

Proof. Since K is compact there exist functions $f_1, \cdots, f_n$ in $\mathfrak{A}_1$ such that $K \subset \bigcup_{i=1}^n W_{f_i}$ but $\bigcup_{i=1}^n W_{f_i} = W_{1/\Sigma_{i=1}^n f_i}$ and $f = 1/n \sum_{i=1}^n f_i \in \mathfrak{A}_1$. too. Now $K \subset \bigcup_{m=1}^\infty \{x \mid f(x) > 1/m\}$ and again a finite union will suffice since K is compact.

Theorem 2. *Let* K *be a compact set in* W *then* $\lim_{n\to\infty} P^n 1_K(x) = 0$ *hence for every measure* $\lim_{n\to\infty} P^{*n} \mu\,(K) = 0$ *and if* μ *is invariant then* $\mu(W) = 0$.

Remark. 1_K is the characteristic function of K, and by a measure we mean, unless otherwise stated, a non-negative σ additive finite measure.

Proof. Let $f \in \mathfrak{A}_1$ such that $1/\delta f \geqq 1_K$ then $P^n 1_K\,(x) \leqq 1/\delta\, P^n f\,(x) \to 0$. Now if μ is a measure $P^{*n}\mu\,(K) = \mu(P^n 1_K) \to 0$ by the Lebesgue Dominated Convergence Theorem. Finally if μ is invariant it vanishes on every compact subset of W. since μ is regular and X is σ compact μ vanishes on W too.

Theorem 3. $P(x, W) = 0$ $x \notin W$.

Proof. Since $P(x, \cdot\,)$ is a regular measure and X is σ compact it is enough to prove that $P(x, K) = 0$ $x \notin W$ and K is a compact subset of W. Use Lemma 1 to find $f \in \mathfrak{A}_1$ with $f \geqq \delta 1_K$ then if $x \notin W$

$$P(x,K) \leqq \frac{1}{\delta} Pf(x) \leqq \frac{1}{\delta} f(x) = 0.$$

If f_1 and f_2 are in $C(X)$ and they differ on W only, then $f_1 - f_2$ is bounded by a multiple of 1_W; hence $Pf_1(x) = pf_2(x)$ if $x \notin W$. It is thus possible to consider P on the complement of W.

DEFINITION. *The operator P is called conservative if $W = \emptyset$ or equivalently $\mathfrak{A}_1 = \{0\}$.*

THEOREM 4. *Let P be conservative. Let $0 \leqq f \in C(X)$ and $Pf \leqq f$. Then $Pf = f$ and $P1_{f>a} = 1_{f>a}$ for every $a \geqq 0$. Thus $P1 = 1$.*

Proof. Put $f_1 = f - Pf$ then $0 \leqq f_1 \in C(X)$ and $g = \sum_{n=0}^{\infty} P^n f_1$ is lower semi-continuous and bounded by f. Put $h = \min(g,1)$ then $Ph \leqq Pg$, $Ph \leqq 1$ or $Ph \leqq h$ also $P^k h \leqq P^k g = \sum_{n=k}^{\infty} P^n f_1 \to 0_{k\to\infty}$. Thus $h \in \mathfrak{A}_1$ and since P is conservative $h = 0$ therefore $g = 0$ and $f = Pf$. In particular $P1 = 1$. Let now $a \geqq 0$ be given. $(f-a)^+ - (f-a)^- = f - a = P(f-a) = P[(f-a)^+] - P[(f-a)^-]$ thus $P[(f-a)^-] \geqq (f-a)^-$. Apply the first part of the theorem to the non-negative continuous function $a - (f-a)^-$ to conclude that $P[(f-a)^-] = (f-a)^-$ and thus $P[(f-a)^+] = (f-a)^+$. Now $P[\min(n(f-a)^+,1)] \leqq \min(n(f-a)^+, 1)$ and again by the first part equality must hold. As $n \to \infty$ $\min(n(f-a)^+,1) \to 1_{f>a}$.

Let A be a measurable set. Define $g_0 = 1_A$ $g_n = \max(g_0, Pg_{n-1})$. Following [4] one sees that the sequence g_n increases to the limit i_A that satisfies $1_A \leqq i_A \leqq 1$ $Pi_A \leqq i_A$ and is the minimal subinvariant function that majorizes 1_A. Note that if A is an open set i_A is lowe semi-continuous function. Thus

LEMMA 6. *Let A be an open set. There exists a function $i_A \in \mathfrak{A}$ such that $1_A \leqq i_A$, $Pi_A \leqq i_A$ and if f is any measurable function with $f \geqq 1_A$, $Pf \leqq f$ then $i_A \leqq f$.*

LEMMA 7. *Let P be a conservative operator. Let $0 \leqq f$ be a measurable function with $Pf \leqq f$. Put $g = \lim P^n f$. Then for every $\delta > 0$ the set $\{x \mid f(x) - g(x) > \delta\}$ does not contain any open set.*

Proof. Assume, to the contrary, that A is an open set and $1_A \leqq 1/\delta(f-g)$. Then $i_A \leqq 1/\delta(f-g)$ and $\lim P^n i_A = 1/\delta \lim P^n(f-g) = 1/\delta(\lim P^n f - g) = 0$ and $i_A \in \mathfrak{A}_1$. Thus A is empty since P is conservative.

THEOREM 8. *Let P be a conservative operator. If $0 \leqq f$ is lower semi-continuous and $Pf \leqq f$ then $Pf < f$ on a set of the first category.*

Proof. Let a, b be rational numbers $0 \leqq b < a$. The set

$$\{x \mid f(x) > a\} \cap \{x \mid Pf(x) \leqq b\}$$

does not contain any open set by Lemma 7. But this set,

$$\{x \mid f(x) > a\} - \{x \mid Pf(x) > b\},$$

is the difference of two open sets and must be contained in the boundary of $\{x \mid Pf(x) > b\}$ which is nowhere dense. (This observation is due to S. Horowitz [see 2]). Thus

$$\{x \mid Pf(x) < f(x)\} = \bigcup [\{x \mid f(x) > a\} \cap \{x \mid Pf(x) \leqq b\} \quad \text{where } 0 \leqq b < a \text{ are rational}]$$

is a set of the first category.

THEOREM 9. *Let P be a conservative operator. Let $0 \leqq f < \infty$ be a lower semi-continuous function then the set $\{x \mid 0 < \sum_{n=0}^{\infty} p^n f(x) < \infty\}$ is of the first category.*

Proof. Put $g = \min(\sum_{n=0}^{\infty} P^n f, 1)$ then g is a lower semi-continuous non-negative function and $Pg \leqq g$. On the set $\{x \mid 0 < \sum P^n f(x) < \infty\}$ the function $g > 0$ and $\lim_{k\to\infty} P^k g = 0$ hence this set is contained in $\bigcup_{n=0}^{\infty}\{x \mid P^{n+1}g(x) < P^n g(x)\}$ which is of the first category by Theorem 8.

REMARK. Theorem 9 was proved in [2] using a different method.

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